Some mathematical properties of the futures market platform

LAIB, Fodil and RADJEF, M.S.
Market Analysis Services, CEVITAL company, LAMOS laboratory, Department of Operations Research, University of Bejaia, Algeria

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M.S. Radjef*and F. Laib†

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Abstract

This is an introductory work to analytical properties of the futures market platform's main parameters. The underlying mechanism of this market structure is formulated into a mathematical dynamical model. Some mathematical properties of traders' positions, their potential and realized wealths, market open interest and average price, are stated and demonstrated.

1 Introduction

The majority of studies on futures markets were conducted from a stochastic perspective where time series were analyzed in order to discover empirical relationships between market phenomena (Chan and Young, 2006; Mandelbrot and Taylor, 1967). In parallel, market analysts and traders use extensively *technical analysis* and *fundamental analysis* to forecast price moves and monitor market trends (CBOT, 1998; Murphy, 1999). In the same optic, Shelton (1997) has suggested an authentic approach based on a game theory model where a rational player (a trader) is playing a game against Nature (the market). He defined the probability triangle showing to the trader the right strategy to play depending on his risk level and the market mood.

However, to respond to more conceptual questions on futures markets, other kind of investigations are needed. For this purpose, Arthur et al. (1996) developed a genetic approach designed to generate the price of a stock financial asset based on heterogeneous agents with different expectations and different strategies. Their approach allowed to understand the band wagon effect and the interaction between technical trading and fundamental trading. Howard (1999) constructed an analytical mathematical model, inspired from the work

*LAMOS laboratory, Department of Operational Research, University of Bejaia, Algeria. Email: ms.radjef@lamos.org

†Market Analysis Services, CEVITAL company, Garidi II, Kouba, Algiers, Algeria. Email: fodil.laib@cevital.com (corresponding author)

of Arthur et al. The model of Howard is basically an evolutionary system generating the price of a stock asset, where some traders are establishing their decisions on technical signals and the others use fundamental signals. At each step of the game, some traders may migrate from one set to the other, influenced by financial results of the other set. The Santa Fe Stock Market Simulator (LeBaron et al., 1999) is a typical computer model of the stock market allowing to carry out simulations and tests the effects of different scenarios on the price behavior (Palmer et al., 1994).

On the other hand, computerized models of futures market platforms are running on almost all brokerage houses systems and commission houses platforms, helping to track traders' positions and monitor their profits and losses on a daily basis. Computerized models have offered to market participants what they need: speed, accuracy and large scale treatment; however, these models are not showing the analytical relationships existing among different market parameters like the link between open interest and average market price. The scientific literature lacks these kind of studies.

By contrast, our work differs from others in that it looks to the futures market platform from a pure mathematical point of view and attempts to establish exact analytical relationships between its components. Furthermore, a mathematical investigation looking in-depth of each phenomena and establishing exact analytical functions between relevant components could be a necessary step to realize new research advances in the field of futures markets price equilibrium understanding. Our investigation is intended to be a contribution to this subject.

The remainder of this study is organized in two sections. The next section describes the underlying mathematical model of a futures market platform and outlines its most important parameters like transactional prices and quantities, traders' states and their update process: At instant $t_j \in \mathbb{T}$, each trader $i \in \mathcal{N}$ is characterized by his position $y_i(t_j)$, his average price $x_i(t_j)$, his potential wealth $w_i(t_j)$, his realized wealth $W_i(t_j)$, and his total wealth $J_i(t_j)$. The market as a whole is characterized by the instantaneous transactional price and quantity, $(p(t_j), q(t_j))$, the open interest $y(t_j)$ and the market average price $\bar{p}(t_j)$ measures. The last section presents our main findings which are analytical relationships between the above mathematical measures of the futures market model. One of the properties on the open interest change seems to have an interesting practical interpretation for market analysis purposes. Finally, to demonstrate these properties, we make use of the *condition function* defined in the appendix.

2 Mathematical formulation of the futures market mechanism

We consider a set of traders, $\mathcal{N} = \{1, \dots, n\}$, tracking a particular futures contract, with a life duration T. Each trader $i \in \mathcal{N}$ is constantly observing the market via the news they receive from different sources of information allowing him to assess the supply and demand levels. Based on these news and their market experience, their needs and their financial capabilities and strategies, traders establish orders, $\mathbf{u}_i$, and send them to the market platform as shown in figure 1.

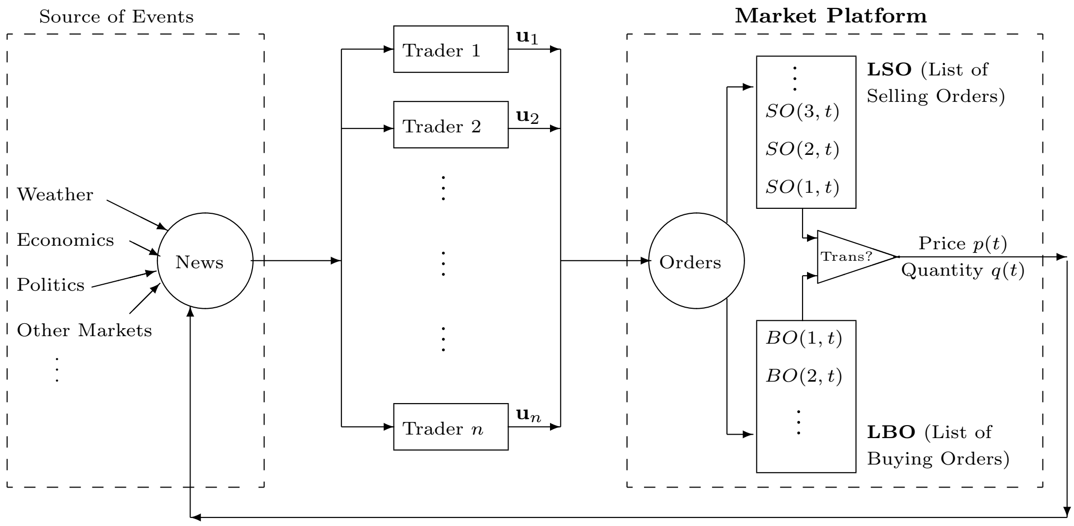


Figure 1: Traders sending their orders to the market platform

The orders are directed either to the List of Selling Orders (LSO) or the List of Buying Orders (LBO) depending on their type. The orders in both lists are instantly sorted in such a way that the best sale order is always in the top of the LSO and the best buy order is in the top of the LBO. At each instant, an attempt is made to generate a transaction between the best buy order with the best sell order.

We assume that the daily market sessions of the futures contract, since the first trading day until expiration day, are grouped into a compacted interval $[0, T]$ which is discretized into a set of instants

$$\mathbb{T} = \{t_0, \ldots, t_m\}, \text{ with } t_0 = 0,\ t_m = T,\ t_j = t_{j-1} + h,\ \ j = 1, \ldots, m,$$

where h is the discretization pace. At instant t_j, at most one order can be received and treated. If an order is received at instant t_j, then it will be directed to the corresponding list of orders, sorted in that list, then an attempt to generate a transaction follows; all these four sub-vents are happening during the same instant t_j.

2.1 Price fixation

The order $\mathbf{u}_i(t_j)$ send to the market platform by trader $i \in \mathcal{N}$ at instant $t_j \in \mathbb{T}$, has the following form:

$$\mathbf{u}_i(t_j) = (u_{i1}(t_j),\ u_{i2}(t_j)) ,$$

where u_{i1} is the *ask* price in case of a sale order, or the *bid* price in case of a buy order, thus $u_{i1} \in \mathbb{R}^+$; and u_{i2} is the number of contracts to sell in case of a sale order, or the quantity to buy in case of buy order[1]. In case of a sale order, we add conventionally a minus sign to distinguish it from a buy order, therefore in the general case $u_{i2} \in \mathbb{Z}$.

At instant t_j, the LSO and LBO display the following status

$$\begin{aligned} SO(1, t_j) \quad &\equiv \mathbf{u}_s(\xi_s) = (u_{s1}(\xi_s),\ u_{s2}(\xi_s)) ; \\ BO(1, t_j) \quad &\equiv \mathbf{u}_b(\xi_b) = (u_{b1}(\xi_b),\ u_{b2}(\xi_b)) ; \end{aligned}$$

that is, the best sale order is $\mathbf{u}_s(\xi_s)$ issued by trader s at instant $\xi_s \leq t_j$; and the best buy order is $\mathbf{u}_b(\xi_b)$ issued by trader b at instant $\xi_b \leq t_j$. A transaction will occur at instant t_j if $u_{s1}(\xi_s) \leq u_{b1}(\xi_b)$, and $u_{s2}(\xi_s) > 0$ and $u_{b2}(\xi_b) > 0$ simultaneously. In this case, the transactional price, $p(t_j)$, will be

$$p(t_j) = \begin{cases} u_{b1}(\xi_b), & if\ \xi_s < \xi_b, \\ u_{s1}(\xi_s), & if\ \xi_b < \xi_s. \end{cases} \tag{1}$$

This price is determined in this way because an advantage is given to the trader who issued his order first. The number of contracts $q(t_j)$ sold by trader s to trader b in this transaction will be

$$q(t_j) = min\{u_{b2}(\xi_b); |u_{s2}(\xi_s)|\}. \tag{2}$$

Otherwise, no transaction will take place at instant t_j, and we set

$$p(t_j) = p(t_{j-1}) \quad \text{and} \quad q(t_j) = 0. \tag{3}$$

If a transaction has occurred at instant t_j, then t_j is a transactional time, otherwise it is a non-transactional time.

2.2 States of the traders

The trading activity of futures contracts starts at instant t_0 and finishes at t_m. At each instant $t_j \in \mathbb{T}$, the state of each trader can be described by the following components:

- $y_i(t_j)$: is the position of trader i, representing the number of contracts he has bought or sold.

[1] A third component, $u_{i3}(t_j)$, may be added to the order $\mathbf{u}_i(t_j)$ in case of a cancelling order to show which previous order of trader i to cancel.

- $x_i(t_j)$: is the average price of the position $y_i(t_j)$ of trader i.
- $w_i(t_j)$: is the potential wealth (profit or loss) of trader i at instant t_j. It represents the amount of money that he would gain or loss if he closes his position at the current instant t_j. This amount is the difference between the real worth of his position and its current worth value, that is

$$w_i(t_j) = y_i(t_j)\,[p(t_j) - x_i(t_j)]\,. \tag{4}$$

- $W_i(t_j)$: is the realized, or closed, wealth (profit or loss) of trader i since the beginning of the game until instant t_j. The component $W_i(t_j)$ is updated only when trader i closes entirely, or partly, his position. If, at instant t_j, he closes $|d(t_j)|$ contracts from his old position then his accumulated realized wealth at instant t_j will be

$$W_i(t_j) = W_i(t_{j-1}) + d(t_j)\,[p(t_j) - x_i(t_{j-1})]. \tag{5}$$

- $J_i(t_j)$: is the total wealth of trader i at instant t_j, defined by

$$J_i(t) = J_i^0 + W_i(t) + w_i(t), \tag{6}$$

  where J_i^0 is the initial wealth of trader i, i.e. the amount of cash he possesses at the beginning of the game.

We set J^0 as the global wealth of all the traders:

$$J^0 = \sum_{i=1}^{n} J_i^0. \tag{7}$$

At the starting time t_0, all the components of each trader are flat, that is

$$x_i(t_0) = y_i(t_0) = w_i(t_0) = W_i(t_0) = 0, \quad i = 1, \dots, n.$$

2.3 Updating traders' states

Consider a step forward in the trading process passing from instant t_{j-1} to t_j, and let's study, in paragraphs 2.3.1 and 2.3.2 respectively, the two possible cases: 'no transaction has occurred' and a 'transaction has occurred' at instant t_j. We are going to deal with some mathematical details as they are needed in the subsequent section for demonstrating some mathematical properties.

Note 2.1. *In order to simplify further our notations and avoid lengthy expressions, we drop the letter t_j when no confusion is possible, hence we set*

$$x_i \equiv x_i(t_j), \quad y_i \equiv y_i(t_j), \quad W_i \equiv W_i(t_j), \quad w_i \equiv w_i(t_j), \quad J_i \equiv J_i(t_j).$$

To make reference to the state of any dynamical variable at the prior instant t_{j-1} we use instead the apostrophe notation ('), that is

$$\begin{array}{lll} p' \equiv p(t_{j-1}), & x_i' \equiv x_i(t_{j-1}), & y_i' \equiv y_i(t_{j-1}), \\ W_i' \equiv W_i(t_{j-1}), & w_i' \equiv w_i(t_{j-1}), & J_i' \equiv J_i(t_{j-1}). \end{array}$$

These notations will be used interchangeably.

2.3.1 Case of no transaction

If no transaction has occurred at instant t_j, then relation (3) will hold, and all the components of each trader will remain unchanged, that is for every $i \in \mathcal{N}$ we have the following:

$$y_i = y_i', \quad x_i = x_i', \quad W_i = W_i', \tag{8a}$$
$$w_i = w_i', \quad J_i = J_i'. \tag{8b}$$

2.3.2 Case where a transaction has occurred

If instant t_j is a transactional time, then a transaction has occurred between a buyer b and a seller s, exchanging q contracts. In this event, an update of the price and the traders' components is necessary. The transactional price, p, and quantity, q, are given by (1) and (2) respectively.

The update of traders' components is conducted in three steps: Step 0 below shows how to update the components of all traders except the buyer and the seller; Step 1 and Step 2 updates the components of the buyer and the seller respectively.

Step 0 : updating all traders' components except the buyer's and the seller's

All the traders other than the buyer b and the seller s, will only update their potential wealth, in other words, for traders $i \in \mathcal{N} \setminus \{b, s\}$ formula (8a) will apply, but their potential wealth component w_i will evolve with time because the price has changed

$$w_i = y_i\,(p - x_i), \qquad i \in \mathcal{N} \setminus \{b, s\}. \tag{9}$$

Obviously, for these traders, their total wealth component J_i, given by (6), should also be recalculated because it depends on w_i.

Step 1: updating the buyer's components

The buyer b has bought q new contracts during this transactional time t_j, his current position y_b will be

$$y_b = y_b' + q. \tag{10}$$

Since he had added new contracts to his old position, the average price x_b of his new position should be updated. However, this update will depend on the value of his previous position y_b'. Below, we examine the four possible cases, 1-i to 1-iv, corresponding respectively to i) $y_b' \geq 0$, ii) $-q < y_b' < 0$, iii) $y_b' = -q$, and iv) $y_b' < -q$. In each case, we determine the analytical expressions of x_b, w_b and W_b.

Step 1 - case i: when $\mathbf{y'_b \geq 0}$. In this case, his new average price x_b on his new position will be

$$x_b = \frac{y'_b x'_b + qp}{y'_b + q}. \tag{11}$$

In this case, his realized wealth will remain unchanged because he has not closed any contract of his old position, thus

$$W_b = W'_b. \tag{12}$$

His potential wealth w_b should be updated because the price has moved from p' to p, that is

$$w_b = y_b(p - x_b). \tag{13}$$

Substituting (10) and (11) in (13) we obtain

$$w_b = y'_b(p - x'_b). \tag{14}$$

Step 1 - case ii: when $\mathbf{-q < y'_b < 0}$. In this case, at instant t_j, he bought q new contracts with a price p. This buying operation can be viewed as two consecutive buying operations:

a) he had bought $|y'_b|$ contracts with a price p, then

b) he bought $q - |y'_b|$ contracts with a price p.

When he executed operation a) he had closed his short position y'_b that he had sold before with a price x'_b, and realized a net profit or loss equal to $|y'_b|(x'_b - p)$. Adding this amount to the old realized wealth W'_b, the new realized wealth will become

$$W_b = W'_b + |y'_b|(x'_b - p) = W'_b + y'_b(p - x'_b).$$

When he executed operation b), he had acquired a long position $y_b = q - |y'_b| = q + y'_b$ with a price $x_b = p$ and the potential wealth of this position is $w_b = y_b(p - x_b) = 0$. This is true because the new position $y_b = q - |y'_b|$ was established at the current price p, therefore it has not yet any potential wealth.

Step 1 - case iii: when $\mathbf{y'_b = -q}$. In this case, when he bought the q new contracts, he had closed entirely his short position, hence he realized a net profit or loss equal to $|y'_b|(x'_b - p)$. Adding this amount to his previous realized wealth, will yield

$$W_b = W'_b + |y'_b|(x'_b - p) = W'_b - q(p - x'_b).$$

In this case, his new position is $y_b = y'_b + q = 0$, thus we consider its average price as $x_b = 0$, having a zero potential wealth, $w_b = y_b(p - x_b) = 0$.

Step 1 - case iv: when $\mathbf{y'_b < -q}$. In this case, when he bought the q new contracts, he had closed q contracts in his old short position, hence he realized a net profit or loss equal to $q(x'_b - p)$. Adding this amount to his previous realized wealth W'_b will result in

$$W_b = W'_b - q(p - x'_b).$$

After this operation, it will remain $y_b = y'_b + q < 0$ contracts in the possession of the buyer. This is a part of his old position that he had sold with an average price x'_b. As these contracts are still in his hand at instant t_j, hence $x_b = x'_b$, and the potential wealth of this position is $w_b = y_b(p - x_b) = (y'_b + q)(p - x'_b)$.

Summary of step 1: In order to write on a single line the functions x_b, W_b, and w_b of the four cases 1-i to 1-iv, we will use the condition function (see appendix) formulation as shown below

$$x_b = \frac{y'_b x'_b + qp}{y'_b + q}\, 1_{[y'_b \geq 0]} + p\, 1_{[-q < y'_b < 0]} + x'_b\, 1_{[y'_b < -q]}; \tag{15}$$

$$W_b = W'_b + (p - x'_b)\left(y'_b\, 1_{[-q < y'_b < 0]} - q\, 1_{[y'_b \leq -q]}\right). \tag{16}$$

However, we have showed that in both cases 1-ii and 1-iii the potential wealth $w_b = 0$. In the remaining cases 1-i and 1-iv, we know that $w_b \neq 0$, hence we can assert that the potential wealth w_b can be written as

$$w_b = (p - x'_b)\left(y'_b\, 1_{[y'_b \geq 0]} + (y'_b + q)\, 1_{[y'_b < -q]}\right). \tag{17}$$

Step 2: updating the seller's components

After selling q contracts, the position of the seller s should be

$$y_s = y'_s - q. \tag{18}$$

Below we examine the four possible cases, 2-i to 2-iv, corresponding respectively to i) $y'_s \leq 0$, ii) $0 < y'_s < q$, iii) $y'_s = q$, and iv) $y'_s > q$. In each case, we determine the analytical expressions of x_s, w_s and W_s.

Step 2 - case i: when $\mathbf{y'_s \leq 0}$. In this case, his new average price x_s on his new position, $y_s = y'_s - q$, will be

$$x_s = \frac{y'_s x'_s - qp}{y'_s - q}. \tag{19}$$

His realized wealth will remain unchanged because he has not closed any contract from his old position, thus

$$W_s = W'_s. \tag{20}$$

His potential wealth, w_s, should be updated due to the price move from p' to p, that is

$$w_s = y_s(p - x_s). \tag{21}$$

Substituting (18) and (19) in (21), we obtain

$$w_s = y'_s(p - x'_s). \tag{22}$$

Step 2 - case ii: when $\mathbf{0 < y'_s < q}$. In this case, the action of the seller can be viewed as two consecutive selling operations:

a) he had sold y'_s contracts with a price p, then

b) he sold $q - y'_s$ contracts with a price p.

When he executed operation a) he had closed his long position y'_s that he had bought before with a price x'_s, and realized a net profit or loss equal to $y'_s(p - x'_s)$. Adding this amount to the old realized wealth W'_s, will yield the new realized wealth

$$W_s = W'_s + y'_s(p - x'_s).$$

When he executed operation b), he had acquired a short position $y_s = -(q - y'_s) = y'_s - q$, with a price $x_s = p$, and the potential wealth of this position is $w_s = y_s(p - x_s) = 0$.

Step 2 - case iii: when $\mathbf{y'_s = q}$. In this case, he had closed entirely his long position, hence he realized a net profit or loss equal to $y'_s(p - x'_s)$. Adding this amount to his previous realized wealth will yield

$$W_s = W'_s + y'_s(p - x'_s) = W'_s + q(p - x'_s).$$

In this case, his new position $y_s = y'_s - q = 0$, thus we consider its average price as $x_s = 0$, and $w_s = y_s(p - x_s) = 0$.

Step 2 - case iv: when $\mathbf{y'_s > q}$. In this case, he had closed q contracts in his old long position, hence he realized a net profit or loss equal to $q(p - x'_s)$. Adding this amount to his previous realized wealth W'_b will result in

$$W_s = W'_s + q(p - x'_s).$$

After this operation, it will remain $y_s = y'_s - q > 0$ contracts in the possession of the seller. This is a part of his old position that he had bought with an average price x'_s. As these contracts are still in his hand at instant t_j, hence $x_s = x'_s$, and the potential wealth of this position is $w_s = y_s(p - x_s) = (y'_s - q)(p - x'_s)$.

Summary of step 2: In order to write on a single line the functions x_s, W_s, and w_s of the four cases 2-i to 2-iv, we will use the condition function formulation as shown below

$$x_s = \frac{y'_s x'_s - qp}{y'_s - q} 1_{[y'_s \le 0]} + p\,1_{[0<y'_s<q]} + x'_s\,1_{[y'_s>q]}; \tag{23}$$

$$W_s = W'_s + (p - x'_s)\left(y'_s\,1_{[0<y'_s<-q]} + q\,1_{[y'_s\ge q]}\right). \tag{24}$$

However, we have showed that in both cases 2-ii and 2-iii that $w_s = 0$. In the remaining cases 2-i and 2-iv, we know that $w_s \neq 0$, hence we can assert that the potential wealth w_s can be written as

$$w_s = (p - x'_s)\left(y'_s\, 1_{[y'_s \leq 0]} + (y'_s - q)\, 1_{[y'_s > q]}\right). \tag{25}$$

3 Some mathematical properties

Two well known properties of a futures market are the following

$$\sum_{i=1}^{n} y_i(t_j) = 0, \quad \text{and} \quad \sum_{i=1}^{n} J_i(t_j) = J^0, \quad \forall\, t_j \in \mathbb{T}. \tag{26}$$

The first result follows directly from (10) and (18) since for every transaction there is a buyer and a seller. The second result reflects the fact that total wealth of all the traders is constant and that what was lost by some traders is gained by others.

Hereafter, we present three classes of new properties.

3.1 Some properties of traders' components

We show herein that the state variables w_i, W_i and J_i, of trader i at instant t_j, can be identified by knowing only their values at the prior instant t_{j-1}, the market price p and the transactional quantity q, of the current transaction, if any.

Property 3.1. *$\forall\, t_j \in \mathbb{T}$, the potential wealth w_i of trader i, defined by relation (4), can be written in the following form*

$$w_i = w'_i + y'_i(p - p'), \tag{27}$$

for every trader $i \in \mathcal{N}$, except if $i = b$ and $y_b < 0$, or if $i = s$ and $y_s > 0$.

Proof. We will prove this property case by case.
a) Case where $i \in \mathcal{N} \setminus \{b,\, s\}$. At instant t_j, we know that $x_i = x'_i$ and $y_i = y'_i$, therefore,

$$\begin{aligned} w_i &= y_i(p - x_i) = y'_i(p - x'_i) \\ &= y'_i(p - p') + y'_i(p' - x'_i) = y'_i(p - p') + w'_i. \end{aligned}$$

b) Case where $i = b$. If $y'_b \geq 0$, hence we should be in the case 1-i of paragraph 2.3.2, then from (14) and following the same reasoning than case a) above, starting from the second line, we show readily this result.

c) Case where $i = s$. If $y'_s \leq 0$, hence we should be in the case 2-i of paragraph 2.3.2, then from (22) and following the same reasoning than case a) above, starting from the second line, we show this result.

Property 3.2. *$\forall\, t_j \in \mathbb{T}$ and $\forall\, i \in \mathcal{N}$, the total wealth, J_i, given by relation (6), can be expressed in terms of J_i' in the following way*

$$J_i = J_i' + y_i'(p - p'). \tag{28}$$

Proof. We will prove this property case by case.
a) For every $i \in \mathcal{N}\backslash\{b,\, s\}$, we know that relations (8-a) and (27) apply, therefore we can write the total wealth J_i defined by (6) as follows

$$J_i \;=\; J_i^0 + W_i + w_i = J_i^0 + W_i' + w_i' + y_i'(p - p') = J_i' + y_i'(p - p').$$

b) If $i = b$, we make use of formulas (16) and (17) in the below development

$$\begin{aligned}
J_b &= J_b^0 + W_b + w_b \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b'\, 1_{[-q<y_b'<0]} - q\, 1_{[y_b'\le -q]}\right) + (p - x_b')\left(y_b'\, 1_{[y_b'\ge 0]} + y_b\, 1_{[y_b'<-q]}\right) \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b'\, 1_{[-q<y_b'<0]} - q\, 1_{[y_b'\le -q]} + y_b'\, 1_{[y_b'\ge 0]} + y_b\, 1_{[y_b'<-q]}\right) \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b'\, 1_{[y_b'>-q]} - q\, 1_{[y_b'\le -q]} + y_b\, 1_{[y_b'<-q]}\right) \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b'\, 1_{[y_b'>-q]} - q\, 1_{[y_b'\le -q]} + y_b\left[1_{[y_b'\le -q]} - 1_{[y_b'=-q]}\right]\right) \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b'\, 1_{[y_b'>-q]} + (y_b - q)\, 1_{[y_b'\le -q]} - y_b\, 1_{[y_b'=-q]}\right);
\end{aligned}$$

but we already know that $1_{[y_b'=-q]} = 1$ if only if $y_b' = -q$, in this event, $y_b = y_b' + q = 0$, hence $y_b\, 1_{[y_b'=-q]} = 0$, is always true. Now we resume the last expression of J_b, after erasing this zero term, we obtain

$$\begin{aligned}
J_b &= J_b^0 + W_b' + (p - x_b')\left(y_b'\, 1_{[y_b'>-q]} + (y_b - q)\, 1_{[y_b'\le -q]}\right); \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b'\, 1_{[y_b'>-q]} + (y_b' + q - q)\, 1_{[y_b'\le -q]}\right) \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b'\, 1_{[y_b'>-q]} + y_b'\, 1_{[y_b'\le -q]}\right) \\
&= J_b^0 + W_b' + y_b'(p - x_b') \\
&= J_b^0 + W_b' + y_b'\left[(p - p') + (p' - x_b')\right] \\
&= J_b^0 + W_b' + y_b'(p' - x_b') + y_b'(p - p') \\
&= J_b^0 + W_b' + w_b' + y_b'(p - p') \\
&= J_b' + y_b'(p - p').
\end{aligned}$$

c) If $i = s$, then following the same approach then case b) above, we can show easily that

$$J_s = J_s' + y_s'(p - p').$$

Hence, relation (28) holds true for all traders and in all cases.

Remark 3.1. *Consider the summation of (28) on all traders,*

$$\sum_{i=1}^{n} J_i = \sum_{i=1}^{n} J_i' + (p - p') \sum_{i=1}^{n} y_i.$$

Since $\sum_{i=1}^{n} y_i = 0$*, then* $\sum_{i=1}^{n} J_i = \sum_{i=1}^{n} J_i'$ *for all* $t_j \in \mathbb{T}$*, i.e. the sum of the wealths of all traders is constant in time. This confirms that property 3.2 is not in disagreement with earlier established results on futures markets (the second term of (26)).*

Remark 3.2. *Assuming that time* t *is continuous in the interval* $[0, T]$*, then total wealth of trader* i *can be described by the following differential equation*

$$\dot{J}_i(t) = y_i(t)\,\dot{p}(t), \quad i = 1, \ldots, n.$$

Property 3.3. $\forall\, t_j \in \mathbb{T}$ *and* $\forall i \in \mathcal{N}$*, the realized wealth* W_i *can be written as*

$$W_i = W_i' + w_i' + y_i'(p - p') - w_i. \tag{29}$$

Proof. If relation (6) was applied at instant t_{j-1}, it would yield

$$J_i' = J_i^0 + W_i' + w_i'.$$

On the other hand, from (6) we can extract the expression of W_i as shown below

$$W_i = J_i - J_i^0 - w_i.$$

Now substituting the term J_i by its expression given in (28) will result in

$$\begin{aligned} W_i &= [J_i' + y_i'(p - p')] - J_i^0 - w_i \\ &= \left[\left(J_i^0 + W_i' + w_i'\right) + y_i'(p - p')\right] - J_i^0 - w_i \\ &= W_i' + w_i' + y_i'(p - p') - w_i. \end{aligned}$$

Remark 3.3. *In case of* $i \in \mathcal{N} \setminus \{b, s\}$*, or* $i = b$ *and* $y_b' \geq 0$*, or* $i = s$ *and* $y_s' \leq 0$*, then we know from section 2.3 that*

$$W_i = W_i',$$

and the remaining part of the right-hand-side of (29) is equal to zero, i.e.

$$w_i' + y_i'(p - p') - w_i = 0,$$

due to property 3.1.

3.2 Some properties of the open interest

The open interest measure, $y(t_j)$, is a popular concept in futures markets. Stated in simple terms, it represents the number of contracts held by traders with *long* positions at instant t_j, which is also equal to the absolute number of contracts held by traders with *short* positions.

Definition 3.1. *The open interest measure, $y(t_j)$, at instant $t_j \in \mathbb{T}$, can be described analytically by*

$$y(t_j) \;=\; \sum_{i=1}^{n} y_i(t_j)\, 1_{[y_i(t_j)>0]} \;=\; -\sum_{i=1}^{n} y_i(t_j)\, 1_{[y_i(t_j)<0]}. \;\square$$

Hereafter, the apostrophe notation (') will apply for y, i.e. the time parameter letter t_j will be dropped in the expression of $y(t_j)$ when no confusion is possible, and the previous state, $y(t_{j-1})$, will be denoted by y'.

Property 3.4. *At an instant $t_j \in \mathbb{T}$, the open interest y can be calculated in the following way*

$$y = y' + A - B, \tag{30}$$

where

$$A \equiv A(t_j) \;=\; q\, 1_{[y_b'>-q]} + y_b'\, 1_{[-q<y_b'\leq 0]}, \tag{31}$$

$$B \equiv B(t_j) \;=\; q\, 1_{[y_s'>q]} + y_s'\, 1_{[0<y_s'\leq q]}. \tag{32}$$

i.e. y depends only on the transactional quantity q and the state of the system at the previous instant t_{j-1}. The amount $A(t_j)$ represents the number of contracts added by the buyer to the open interest, and $B(t_j)$ indicates the number of contracts deducted by the seller from the open interest.

Proof. We have

$$\begin{aligned} y &= \sum_{i\in\mathcal{N}} y_i\, 1_{[y_i>0]} \\ &= y_b\, 1_{[y_b>0]} + y_s\, 1_{[y_s>0]} + \sum_{i\in\mathcal{N}\setminus\{b,s\}} y_i\, 1_{[y_i>0]} \\ &= y_b\, 1_{[y_b>0]} + y_s\, 1_{[y_s>0]} + \sum_{i\in\mathcal{N}\setminus\{b,s\}} y_i'\, 1_{[y_i'>0]}, \end{aligned}$$

we write this as

$$y = Q_1 + Q_2 + \sum_{i\in\mathcal{N}\setminus\{b,s\}} y_i'\, 1_{[y_i'>0]}; \tag{33}$$

where

$$\begin{aligned} Q_1 &= y_b\, 1_{[y_b>0]} = (y_b' + q)\, 1_{[y_b'+q>0]} = (y_b' + q)\, 1_{[y_b'>-q]} \\ &= y_b'\, 1_{[y_b'>-q]} + q 1_{[y_b'>-q]} = y_b' \left(1_{[y_b'>0]} + 1_{[-q<y_b'\leq 0]}\right) + q 1_{[y_b'>-q]} \\ &= y_b'\, 1_{[y_b'>0]} + A; \end{aligned}$$

and

$$
\begin{aligned}
Q_2 &= y_s\, 1_{[y_s>0]} = (y'_s - q)\ 1_{[y'_s - q>0]} = (y'_s - q)\ 1_{[y'_s>q]} \\
&= (y'_s - q)\ \left(1_{[y'_s>q]} + 1_{[0<y'_s\leq q]} -\ 1_{[0<y'_s\leq q]}\right) = (y'_s - q)\ \left(1_{[y'_s>0]} -\ 1_{[0<y'_s\leq q]}\right) \\
&= y'_s 1_{[y'_s>0]} - q\, 1_{[y'_s>0]} - y'_s\, 1_{[0<y'_s\leq q]} + q\, 1_{[0<y'_s\leq q]} \\
&= y'_s\, 1_{[y'_s>0]} - q\left(1_{[y'_s>0]} -\ 1_{[0<y'_s\leq q]}\right) - y'_s\, 1_{[0<y'_s\leq q]} \\
&= y'_s\, 1_{[y'_s>0]} - q\, 1_{[y'_s>q]} - y'_s\, 1_{[0<y'_s\leq q]} \\
&= y'_s\, 1_{[y'_s>0]} - B.
\end{aligned}
$$

Substituting Q_1 and Q_2 in formula (33), we obtain

$$
\begin{aligned}
y &= y'_b\, 1_{[y'_b>0]} + A + y'_s\, 1_{[y'_s>0]} - B + \sum_{i\in\mathcal{N}\ \setminus\{b,s\}} y'_i\, 1_{[y'_i>0]} \\
&= A - B + \sum_{i\in\mathcal{N}} y'_i\, 1_{[y'_i>0]} \\
&= y' + A - B.
\end{aligned}
$$

Property 3.5. *$\forall\, t_j \in \mathbb{T}$, the open interest $y(t_j)$ could be calculated by*

$$
y(t_j) = \sum_{k=0}^{j} \left[A(t_k) - B(t_k)\right]. \tag{34}
$$

Proof. By definition, we know that $y(t_0) = 0$ because $y_i(t_0) = 0$, $\forall i \in \mathcal{N}$, so (34) holds for t_0. Now, assuming that at instant t_{j-1} relation (34) holds, that is,

$$
y(t_{j-1}) = \sum_{k=0}^{j-1} \left[A(t_k) - B(t_k)\right],
$$

hence

$$
y(t_j) = y(t_{j-1}) + A(t_j) - B(t_j) = \sum_{k=0}^{j-1} \left[A(t_k) - B(t_k)\right] + A(t_j) - B(t_j) = \sum_{k=0}^{j} \left[A(t_k) - B(t_k)\right].
$$

□

The value and sign of the change in the open interest measure are monitored continuously by traders and analysts as it helps them assessing the behavior of the market and forecasting its future move.

Property 3.6. *Consider the change in open interest, $\Delta y(t_j)$, at a transactional time t_j, defined by*

$$
\Delta y = y - y'.
$$

For a specified value of q, and allowing the values of y'_b and y'_s to vary over the set of integer numbers, then the values and signs of Δy in each case are given in tables 1a and 1b respectively.

	$y'_b \leq -q$	$-q < y'_b \leq 0$	$y'_b > 0$
$y'_s > q$	$-q$	y'_b	0
$0 < y'_s \leq q$	$-y_s$	$q + y'_b - y'_s$	$q - y'_s$
$y'_s \leq 0$	0	$q + y'_b$	q

(a) Values of Δy

	$y'_b \leq -q$	$-q < y'_b \leq 0$	$y'_b > 0$
$y'_s > q$	< 0	≤ 0	0
$0 < y'_s \leq q$	< 0	any	≥ 0
$y'_s \leq 0$	0	> 0	> 0

(b) Signs of Δy

Table 1: Values and signs of Δy

Proof. Note that if t_j is a non-transactional time, then $y = y'$, therefore $\Delta y = 0$. Thereafter, we are dealing with transactional times only. From (30), we deduce that

$$\Delta y \quad = \quad y - y' = A - B = \left(q\,1_{[y'_b > -q]} + y'_b\,1_{[-q < y'_b \leq 0]}\right) - \left(q\,1_{[y'_s > q]} + y'_s\,1_{[0 < y'_s \leq q]}\right).$$

Table 2 summarizes the calculation for each case: case 1) corresponds to $y'_b > 0$ and $y'_s > q$, case 2) corresponds to $y'_b > 0$ and $0 < y'_s \leq q$, and so on. For each case, we compute the values of A and B, then we calculate the difference $\Delta y = A - B$, and the last column of the table shows the sign of Δy in each case.

case	y'_b	y'_s	$A =$	$B =$	$\Delta y = A - B =$	Sign of Δy
1)	$y'_b > 0$	$y'_s > q$	q	q	0	0
2)		$0 < y'_s \leq q$	q	y'_s	$q - y'_s$	≥ 0
3)		$y'_s \leq 0$	q	0	q	> 0
4)	$-q < y'_b \leq 0$	$y'_s > q$	$q + y'_b$	q	y'_b	≤ 0
5)		$0 < y'_s \leq q$	$q + y'_b$	y'_s	$q + y'_b - y'_s$	any
6)		$y'_s \leq 0$	$q + y'_b$	0	$q + y'_b$	> 0
7)	$y'_b \leq -q$	$y'_s > q$	0	q	$-q$	< 0
8)		$0 < y'_s \leq q$	0	y'_s	$-y'_s$	< 0
9)		$y'_s \leq 0$	0	0	0	0

Table 2: Calculation of Δy

Case 5) of table 2, where $-q < y'_b \leq 0$ and $0 < y'_s \leq q$, necessitates further analysis to determine the sign of Δy. In this case, we know that

$$\Delta y = q + y'_b - y'_s. \tag{35}$$

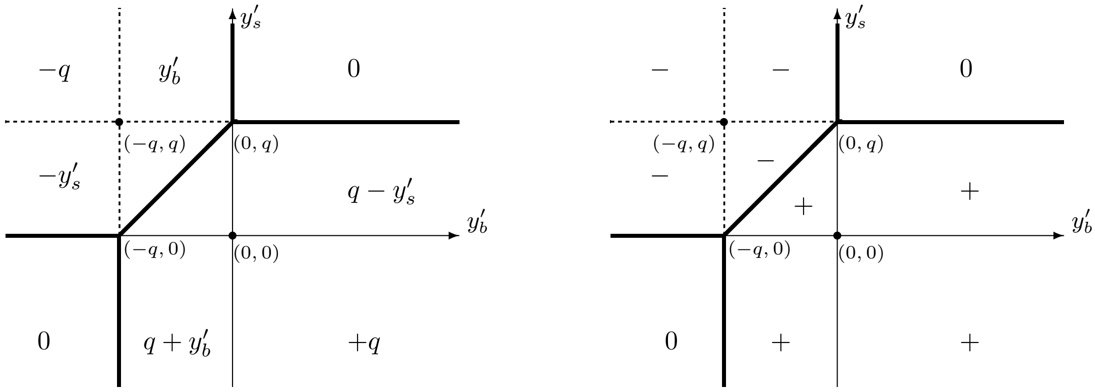
For this case 5), we can show easily that $-2q < y'_b - y'_s < 0$, hence $-q < q + y'_b - y'_s < q$, therefore Δy could be positive, negative or null, depending on the values of y'_b and y'_s; we have the following

- $\Delta y > 0 \;\Rightarrow\; q + y'_b > y'_s$;
- $\Delta y < 0 \;\Rightarrow\; q + y'_b < y'_s$;

- $\Delta y = 0 \quad \Rightarrow \quad q + y'_b = y'_s$.

This completes the proof of this property.

Remark 3.4. *The results of tables 1a and 1b can be further displayed graphically on a 2-dimension space with $(0,0)$ as an origin, the horizontal X-axis representing y'_b versus the vertical Y-axis for y'_s . This is shown in figures 2a and 2b.*



(a) Values of Δy **(b)** Signs of Δy

Figure 2: Values and signs of Δy

Figure 2a shows the values of Δy for each point $(y'_b, y'_s) \in Z^2$. Inside the square delimited by the points $(0,0)$, $(-q,0)$, $(-q,q)$ and $(0,q)$, the value of Δy is calculated by formula (35); this square corresponds to case 5 of table 2. In addition to the two zones where $\Delta y = 0$, all the points belonging to the thick lines correspond also to $\Delta y = 0$.

On the other hand, figure 2b shows the signs of Δy for each point $(y'_b, y'_s) \in Z^2$. Inside the triangle delimited by the points $(0,0)$, $(-q,0)$ and $(0,q)$, the sign of Δy is positive. All the points of the triangle $(-q,0)$, $(-q,q)$, $(0,q)$ correspond to a negative Δy. The points of the common segment $(-q,0)$, $(0,q)$ of these two triangles have $\Delta y = 0$.

Property 3.7. *Assume that M is the biggest number in the set of positive integer numbers (in practice, M stands for $+\infty$). At a transactional time, the probability $\pi(\cdot)$ of the following events are*

$$\begin{aligned}
\pi(\Delta y = 0) &= \frac{1}{2} - \frac{q}{2M}, \\
\pi(\Delta y > 0) &= \frac{1}{4} + \frac{q}{2M}\left(1 + \frac{q}{4M}\right), \\
\pi(\Delta y < 0) &= \frac{1}{4} - \frac{1}{8}\left(\frac{q}{M}\right)^2.
\end{aligned}$$

Proof. Assuming that M is the biggest positive number, then from figure 3 we observe that any couple (y'_b, y'_s) belongs to the square delimited by the points

$(-M,-M)$, $(-M,M)$, (M,M), and $(M,-M)$, having an area $4M^2$ square-units.

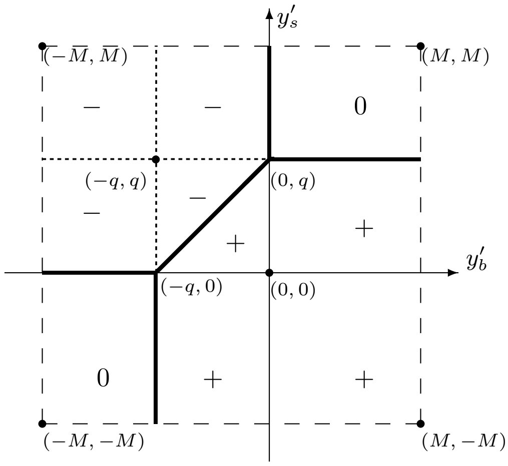


Figure 3: Calculation of $\pi(\Delta y)$

In this square, we have:

- two symmetrical zones where $\Delta y = 0$, with a total area of $2M(M-q)$ square-units, hence

$$\pi(\Delta y = 0) = \frac{2M(M-q)}{4M^2} = \frac{1}{2} - \frac{q}{2M};$$

- one zone where $\Delta y > 0$ formed by four sub-zones: one square of M^2 square-units, two symmetrical rectangles of $2Mq$ square-units, and a triangle of $\frac{q^2}{2}$ square-units; therefore

$$\pi(\Delta y > 0) = \frac{M^2 + 2Mq + \frac{q^2}{2}}{4M^2} = \frac{1}{4} + \frac{q}{2M} + \frac{q^2}{8M^2} = \frac{1}{4} + \frac{q}{2M}\left(1 + \frac{q}{4M}\right);$$

- one zone where $\Delta y < 0$ formed by four sub-zones: one square with $(M-q)^2$ square-units, two symmetrical rectangles with $2Mq$ square-units, and a triangle of $\frac{q^2}{2}$ square-units. We can also consider this zone as being formed by a bigger square $(0,0)$, $(-M,0)$, $(-M,-M)$ and $(0,M)$, having an area of M^2 square-units, from which we deducted the triangle $(0,0)$, $(-q,0)$, (0,q) having an area of $\frac{q^2}{2}$ square-units, thus

$$\pi(\Delta y < 0) = \frac{M^2 - \frac{q^2}{2}}{4M^2} = \frac{1}{4} - \frac{1}{8}\left(\frac{q}{M}\right)^2.$$

Property 3.8. *At a transactional time, assuming that q can vary from 1 to M, then we have the following limits on the probabilities of each event.*

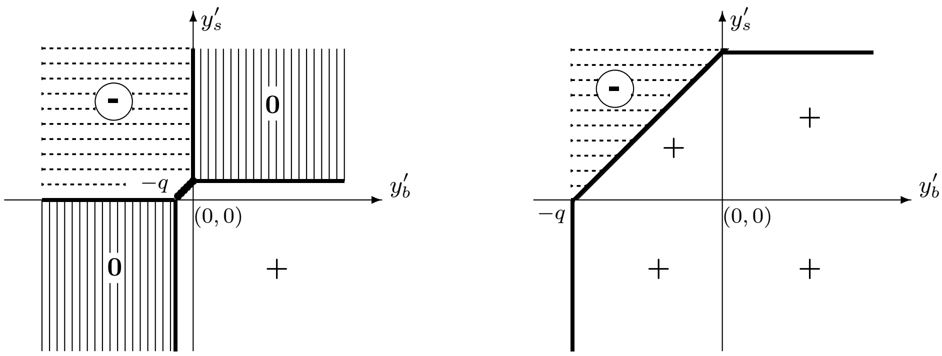
$$\lim_{q\to 1} \pi(\Delta y = 0) = \frac{1}{2}, \qquad \lim_{q\to M} \pi(\Delta y = 0) = 0,$$
$$\lim_{q\to 1} \pi(\Delta y > 0) = \frac{1}{4}, \qquad \lim_{q\to M} \pi(\Delta y > 0) = \frac{7}{8},$$
$$\lim_{q\to 1} \pi(\Delta y < 0) = \frac{1}{4}, \qquad \lim_{q\to M} \pi(\Delta y < 0) = \frac{1}{8}.$$

Proof. Assuming that M is bigger enough ($M \equiv +\infty$), then

$$\lim_{q\to 1} \frac{q}{M} = 0, \text{ and } \lim_{q\to M} \frac{q}{M} = 1.$$

Applying these two limits we show easily property 3.8. □

Graphical visualization: The results of property 3.8 are illustrated graphically on figures 4a-b.



(a) Case where $q \to 1$ **(b)** Case where $q \to \infty$

Figure 4: Limits of $\pi(\Delta y)$

The first figure, 4a, shows the case where q is small enough ($q \to 1$). The dotted area represents the zone where $\Delta y < 0$, the blank area corresponds to $\Delta y > 0$, and the two symmetrical dashed areas illustrate the points (y'_b, y'_s) for which $\Delta y = 0$. From a rough observation, we note that the two dashed zones occupy almost halve of the plane, confirming the fact $\lim_{q\to 1} \pi(\Delta y = 0) = \frac{1}{2}$. Whereas, the blank and dotted zones fill approximately one quarter of the plane for each, hence confirming the limits $\lim_{q\to 1} \pi(\Delta y > 0) = \frac{1}{4}$ and $\lim_{q\to 1} \pi(\Delta y < 0) = \frac{1}{4}$ respectively. We observe also that the blanc zone is slightly larger than the dotted zone, showing that

$$\pi(\Delta y < 0) < \frac{1}{4} < \pi(\Delta y > 0).$$

Figure 4b illustrates the case where $q \to \infty$. We observe readily that the two dashed zones are no longer visible on this plane, hence confirming that $\lim_{q\to M} \pi(\Delta y = 0) = 0$. On the other side, the blanc zone spreads over a greater space, approximately equal to 7/8, proving that $\lim_{q\to M} \pi(\Delta y > 0) = \frac{7}{8}$, and inversely, the dotted zone is smaller than before and occupies only 1/8 confirming that $\lim_{q\to M} \pi(\Delta y < 0) = \frac{1}{8}$.

Contribution to market analysis: Property 3.8 can bring further insight to market analysts. Indeed, after a transaction has occurred, the open interest y could either increase, or decrease or stagnate; this is reflected by the sign of Δy. This change depends on the transactional quantity q, and the buyer's prior position y'_b and the seller's prior position y'_s; all possible cases are given in table 1a. For instance, if the buyer was long or flat before the transaction, i.e. $y'_b \geq 0$, and the seller was short or flat, i.e. $y'_s \leq 0$, then for any value of q, the open interest will increase as a result of this transaction.

If the transactional quantity is small enough ($q \to 1$), then it is more likely that the open interest will stagnate after the transaction rather than increase or decrease, since the event $\Delta y = 0$ has about 50% of chances to occur, whereas the events $\Delta > 0$ and $\Delta y < 0$ have only about 25% of chances for each to occur.

By contrast, if the transactional quantity is big enough, i.e. q is of the same order than y'_b and y'_s, then it is more likely that the open interest will increase; in fact, this should happen in 75% of cases, and the possibility to see open interest decreases is only 25% in this case. Noticeably, in this case, the open interest should not stagnate as the probability of the event $\Delta y = 0$ is almost zero.

3.3 A property of the market average price

Definition 3.2. *We define the* market average price*,* $\bar{p}$*, at instant* t_j *by*

$$\bar{p}(t_j) = \frac{\sum_{k=0}^{j} p(t_k) q(t_k)}{\sum_{k=0}^{j} q(t_k)}, \tag{36}$$

which is simply the weighted average price of all the transactions since the starting time t_0 *until* t_j*.*

The following property links the average price of the market to the open interest and the traders' components.

Property 3.9. *In the special case where*

$$y(t_k) = y(t_{k-1}) + q(t_k), \quad \forall\, k = 0, \ldots, j, \tag{37}$$

then

$$\bar{p}(t_j) = \frac{\sum_{i=1}^{n} x_i(t_j) y_i(t_j)\, 1_{[y_i(t_j)>0]}}{y(t_j)}. \tag{38}$$

That is, formula (38) allows to compute the market average price at instant t_j *using only the knowledge available at this instant.*

Proof. Condition (37) means that, since the instant t_0 till t_j, no trader is closing a part of his old position, i.e. any trader who bought before continues to buy and any trader who sold before continues to sell. In a mathematical form, if s_k and b_k are respectively the seller and buyer at a transactional instant t_k, then

$$y_{s_k}(t_{k-1}) \leq 0, \quad \text{and} \quad y_{b_k}(t_{k-1}) \geq 0, \quad \forall\, k = 0, \dots, j.$$

In this case, the open interest $y(t_k)$ at any instant t_k is growing by the amount of the transactional quantity $q(t_k)$, therefore

$$y(t_j) = y(t_{j_1}) + q(t_j) = \sum_{k=0}^{j} q(t_k).$$

Assuming that (38) holds true at t_{j-1}, that is

$$\bar{p}(t_{j-1}) = \frac{\sum_{k=0}^{j-1} p(t_k)q(t_k)}{\sum_{k=0}^{j-1} q(t_k)} = \frac{\sum_{i=1}^{n} x_i(t_{j-1})y_i(t_{j-1})\, 1_{[y_i(t_{j-1})>0]}}{y(t_{j-1})}; \tag{39}$$

and let's show this remains true at t_j. We already know that

$$\bar{p}(t_j) = \frac{\sum_{k=0}^{j} p(t_k)q(t_k)}{\sum_{k=0}^{j} q(t_k)} = \frac{\sum_{k=0}^{j-1} p(t_k)q(t_k) + p(t_j)q(t_j)}{\sum_{k=0}^{j-1} q(t_k) + q(t_j)} \tag{40}$$

$$= \frac{\sum_{i=1}^{n} x_i(t_{j-1})y_i(t_{j-1})\, 1_{[y_i(t_{j-1})>0]} + p(t_j)q(t_j)}{y(t_{j-1}) + q(t_j)}. \tag{41}$$

We know that when passing from instant t_{j-1} to t_j, the components of all traders will remain the same, except the components of the buyer b and the seller s need to be updated, that is $x_i(t_j) = x_i(t_{j-1})$ and $y_i(t_j) = y_i(t_{j-1})$ for all $i \in \mathcal{N} \setminus \{s, b\}$. This is true in case of a transactional time. In case of a non-transactional time, components of all traders will remain the same. Now resuming the apostrophe notation, we obtain

$$\bar{p} = \frac{\sum_{i \in \mathcal{N}\setminus\{b\}} x_i y_i\, 1_{[y_i>0]} + x'_b y'_b + pq}{y + q}. \tag{42}$$

Note that the components of the seller s do not appear above because we are in the case where $y'_s < 0$, therefore $1_{[y_s>0]} = 0$.

1) If t_j is not a transactional time, i.e. $q = 0$, $y = y'$, $y_b = y'_b$ and $x_b = x'_b$, then from (42) will result

$$\bar{p} = \frac{\sum_{i \in \mathcal{N}} x_i y_i\, 1_{[y_i>0]}}{y} \tag{43}$$

which completes the proof.

2) If t_j is a transactional time, and since $y'_b \geq 0$ then $y_b = y'_b + q > 0 \Rightarrow 1_{[y_b>0]} = 1$ and

$$x_b = \frac{x'_b y'_b + pq}{y_b} \quad \Rightarrow \quad pq = x_b y_b - x'_b y'_b. \tag{44}$$

Substituting pq by $x_b y_b - x'_b y'_b$ in (42) will readily complete the proof.

4 Conclusion and perspectives

Our study showed that a futures market platform has reach analytical properties. We derived the most basic of them, and we believe that many other features remain to be explored and stated in a mathematical framework. A more important issue is to bring practical interpretation of these properties as it was done with property 3.8. In addition, some results need to be generalized, this is the case of property 3.9 on the market average price that need to be extended to the case where condition (37) is no more verified.

On the other hand, the mathematical model of the futures market platform as stated herein has already a theoretical game format, though a discussion on the game equilibrium is lacking. This could be achieved by introducing trading strategies for traders as it was carried out by Arthur et al. (1997) for the stock market. Furthermore, we may write the model in a matrix format in order to simplify notations and present the model in a more compact form. Additionally, the continuous-time version of the model can be considered as pointed out in remark 3.2.

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Appendix: The condition function

The condition function is a new version of the boolean function (zero-one). It allows to summarize, in a single analytical formula, the content and meaning of many statements each of them related to a realization of a particular condition. The condition function is particularly useful when demonstrating mathematical properties involving several cases, as it helps to aggregate all cases in a single relation and performs the mathematical demonstration using only logic arguments, consequently avoiding lengthy and discourse proofs.

Definition 4.1. *The condition function* $1_{[.]}$ *is defined by*

$$1_{[R(a,b)]} = \begin{cases} 1, & \textit{if relation R(a,b) is true}, \\ 0, & \textit{if relation R(a,b) is false}; \end{cases}$$

where $R(a, b)$ *is an algebraic relationship between the two mathematical entities* a *and* b*. (For instance,* R *could be an equality (*$=$*), or inequality (*$\leq$*,* $\geq$*), or inclusion (*$\in$*), etc.)*

Application: The general form of a step-wise function is

$$f(x) = \begin{cases} h(x), & \text{if } x \leq a, \\ g(x), & \text{if } a < x < b, \\ 0, & \text{if } x > b. \end{cases}$$

Using the condition function, the step-wise function can presently be written on a single line:

$$f(x) = h(x)\, 1_{[x \leq a]} + g(x)\, 1_{[a < x \leq b]}$$

Also, the condition function can replace the *min* and *max* functions as shown below

$$\min\{x, y\} = x\, 1_{[x \leq y]} + y\, 1_{[x > y]}, \qquad \max\{x, y\} = x\, 1_{[x \geq y]} + y\, 1_{[x < y]}.$$

Some properties of the condition function: Consider a and b are two real parameters with $a < b$, and x and y are real unknowns, then we have the following

- $1_{[x<a]} + 1_{[x \geq a]} = 1, \qquad 1_{[x \leq a]} = 1_{[x<a]} + 1_{[x=a]},$
- $1_{[x>a]} = 1_{[x>b]} + 1_{[a<x \leq b]}, \qquad 1_{[x>b]} = 1 \;\Rightarrow\; 1_{[x>a]} = 1,$
- $1_{[a<x<b]} = 1_{[x>a]}\, 1_{[x<b]}, \qquad 1_{[x<a]}\, 1_{[x<b]} = 1_{[x<a]},$
- $1_{[x>a]}\, 1_{[x>b]} = 1_{[x>b]}, \qquad 1_{[\,1_{[x>0]}>0]} = 1_{[x>0]},$
- $1_{[y+b\, 1_{[x>0]}>0]} = 1_{[x>0]}\, 1_{[y+b>0]} + 1_{[x \leq 0]}\, 1_{[y>0]},$
- $\min\{0, x\} = x\, 1_{[x \leq 0]}, \qquad \max\{0, x\} = x\, 1_{[x \geq 0]}.$