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Author(s): Stephen A. Buser
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LaPlace Transforms as Present Value Rules: A Note

STEPHEN A. BUSER*

ABSTRACT

The present value equation in finance is shown to be equivalent to the Laplace transformation in mathematics. Based on this observation, the list of known analytic solutions for the present value problem is increased from a handful to more than one hundred. General properties of the Laplace transform are examined as well in light of the newly discovered significance for finance.

ONE OF THE RECURRING problems in finance is to find the present value of a given cash flow $C(t)$ for a given rate of discount r.

$$V(r) = \int_0^\infty e^{-rt} C(t) dt. \tag{1}$$

In standard mathematical jargon, the present value integral, $V(r)$, is referred to as the Laplace transform of the cash flow, $C(t)$. Apart from semantics, there is a substantial practical gain from identifying the generic name of the present value problem: only a handful of analytic solutions have been found for present value problems per se. Yet, thanks to Laplace (1749–1827) and countless others who have followed in his work, more than a hundred solutions exist for the Laplace transformation.[1]

The purpose of this note is to alert the profession to this rich source of present value rules and to illustrate the enhanced problem-solving capabilities of the expanded bag of tricks. In particular, it is shown that assets with cyclical cash flows can have surprisingly simple representations. Present value rules for each of the elementary mathematical functions are identified in Section I. In Section II, general properties of the Laplace transformation are presented and discussed. These properties are then used in Section III to confirm the elementary rules and to show in Section IV how the simple rules can be combined to approximate the present value of complex cash flows.

I. Present Value Rules for the Elementary Functions

Table I is offered as specific motivation for a closer look at the Laplace transformation. Lines 1 and 2 list the customary expressions for the present value of the consol, or level payment stream, and the geometric growth stream, respectively.

* The Ohio State University.

[1] For an extensive list of rules, see Murray R. Spiegel, *Laplace Transforms*, NY: McGraw-Hill, 1965.

Table I

Laplace Transforms/Present Values for the Elementary Functions

Cash Flow $C(t)$	Present Value $V(r)$
1. Constant (1)	$1/r$
2. Geometric (e^{at})	$1/(r-a)$ for $a<r$
3. Arithmetic (t)	$1/r^2$
4. Power (t^n)	$n!/r^{n+1}$
5. Sine [$\sin(t)$]	$1/(1+r^2)$
6. Cosine [$\cos(t)$]	$r/(1+r^2)$

Line 3 shows that an arithmetic growth stream is equivalent to receiving one consol per period in perpetuity (a consol of consols). The rule is not widely used but can be found in many standard sources for present value solutions. It is doubtful, however, that even a specialist would recognize the remaining expressions for even the simplest of cash flows. For example, the rule for the present value of a power sequence (line 4) is not generally known even though it is a natural extension of rules 1 and 3 (special cases $n = 0$ and $n = 1$, respectively). In a similar vein, future generations of security analysts may find it surprising that present value rules for the periodic functions would have escaped attention for so many years despite the profession's general concern for the cyclical behavior of asset prices. Lines 5 and 6 show that present value rules for elementary periodic cash flows are neither less tractible nor more cumbersome than present value rules for elementary noncyclical cash flows. On the contrary, the rules are strikingly simple. At moderate rates of discount, the present value of the standard sine cash flow (unit amplitude) is approximately \$1. The present value of the standard cosine cash flow is approximately equal to the rate of discount.

II. General Properties of the Laplace Transformation

Beyond the simple intuitive appeal of each of the rules in Table I, the collection is interesting in the sense that all six expressions can be derived from a single property of the Laplace transformation. That unifying rule is identified in Table II (line 8) along with other properties of the Laplace transformation that appear to have particular significance for finance.

Line 1 states that the Laplace transformation is a linear operator. Line 2 shows that scaling a cash flow by a geometric growth term is equivalent to a corresponding reduction in the rate of discount. Both rules are readily apparent from the definition of the Laplace transformation as the integral of an exponentially weighted function [Equation (1)].

Line 3 shows the effect of scaling a cash flow by an arithmetic growth term. Readers who are familiar with the Hicks/Macaulay measure of duration (time-weighted present value) should recognize the link to interest rate elasticity that is implied by this rule. To confirm the result, recall that the derivative of the

Table II

The Algebra of Laplace Transforms/Present Values*

	Cash Flow	Transform
1. Linearity	$aC(t) + bD(t)$	$aV(r) + bW(r)$
2. Geometric scaling	$e^{at}C(t)$	$V(r-a)$ for $a < r$
3. Multiplication by t	$tC(t)$	$-V'(r)$
4. Division by t	$C(t)/t$	$\int_r^{\infty} V(x)\,dx$
5. Time shift	$C(a+bt)$ for $t \geq a/b$; 0 for $t < a/b$	$e^{ra/b}(1/b)V(r/b)$
6. Point flow	$C(t)$ for $t = k$; 0 for $t \neq k$	$e^{-rt}C(k)$
7. Finite life	$C(t)$ for $t \leq k$; 0 for $t > k$; with $C(t) = D(t-k)$	$V(r) - e^{-rk}W(r)$
8. Time derivative	$C'(t)$	$rV(r) - C(0)$
9. Integral	$\int_0^t C(x)\,dx$	$V(r)/r$

* In the table, a and b are arbitrary constants and $V(r)$ and $W(r)$ are the transforms of the cash flows $C(t)$ and $D(t)$, respectively.

exponential function, $\exp(-rt)$, taken with respect to r, is simply the function itself scaled by $-t$.

The rule for division by the time index (line 4) is a corollary to line 3 that follows from Leibnitz's rule for the derivative of a definite integral taken with respect to its lower bound.

Line 5 applies the change-of-variable theorem of integral calculus and is particularly useful for evaluating cash flows with altered time schedules (accelerated or deferred.) In addition, rule 5 can be used in conjunction with the trivial rule for the transform of a single payment (line 6) to evaluate flows with finite lives as indicated on line 7.

Line 8 identifies a fundamental linear relationship between Laplace transforms for cash flows and their time derivatives. This property is worthy of special note for two reasons. First, the proof is nontrivial, and that alone sets it apart from the other rules in Table II. Second, the property is a generalization of the customary procedure for solving the present value equation by applying the rule for summing (or integrating) geometric series. To confirm the time-derivative property, note that integration by parts implies that:

$$e^{-rt}C(t) = -r\int e^{-rt}C(t)dt + \int e^{-rt}C'(t)dt. \qquad (2)$$

Rule 8 follows immediately from Equation (2) when we evaluate the integral over the relevant range for the Laplace transform $[0, \infty]$ and impose a standard assumption in present value problems that the marginal present value of the cash flow, $\exp(-rt)C(t)$, vanishes as t gets large.

Line 9 is a corollary to property 8 that follows from Leibnitz's rule for the derivative of a definite integral taken with respect to its upper bound.

III. Applications of the Time-Derivative Property

To confirm that each of the present value rules in Table I can be derived from property II.8, it is convenient to introduce a notational form for the Laplace transformation that allows us to keep track of the underlying cash flow:

$$L[C(t)] = V(r). \tag{3}$$

With this notation, we can restate the time-derivative property as:

$$L[C(t)] = C(0)/r + L[C'(t)]/r. \tag{4}$$

The rule for the consol (I.1) follows trivially from Equation (4) and the observation that for $C(t) = 1$, we have $C(0) = 1$ and $C'(t) = 0$. In turn, the consol rule provides an interesting "myopic" interpretation of the time-derivative rule (line II.8): each asset is valued as if its cash flow were projected at a constant level equal to the current rate plus the present value of the time derivative of the cash flow.

In the case of the geometric cash flow (line I.2), we have $C(t) = \exp(at)$, so that $C(0) = 1$ and $C'(t) = aC(t)$. Hence, by virtue of property II.8, we have:

$$L[\exp(at)] = 1/r + aL[\exp(at)]/r, \tag{5}$$

and rule I.2 follows immediately. Alternatively, we could combine the consol rule with property II.2 to establish the rule for geometric growth.

To derive the rule for arithmetic growth (I.3), we could combine the consol rule (I.1) and property II.3. Or, we can apply property II.8 and note that for $C(t) = t$, we have $C(0) = 0$ and $C'(t) = 1$. More generally, for any power sequence $C(t) = t^n$, we have $C(0) = 0$ and $C'(t) = nt^{n-1}$ so that:

$$L[t^n] = (n/r)L[t^{n-1}]. \tag{6}$$

By repeating this observation n times, we can establish the general rule for the present value of any power cash flow (I.4). Alternatively, we can confirm the result by repeated applications of rule II.3.

To derive present value rules for the simple periodic cash flows (lines I.5 and I.6), we apply property II.8 to both the sine and cosine functions; i.e., because $\sin(0) = 0$, $\cos(0) = 1$, $\sin'(t) = \cos(t)$, and $\cos'(t) = -\sin(t)$, we have:

$$L[\sin(t)] = L[\cos(t)]/r, \tag{7}$$

and

$$L[\cos(t)] = 1/r - L[\sin(t)]/r. \tag{8}$$

We can solve the two equation system by substituting into Equation (7) on the basis of Equation (8) and the desired results follow immediately.

IV. Present Value Rules in Combination

As an illustration of the potential combinations of the rules in Table I and the properties in Table II, consider the case of the lowly k-period annuity. Real world counterparts include fixed-rate mortgages, and the coupon portion of most bond returns, among numerous other examples. We can apply property II.7 with $C(t) = C(t-k) = 1$, so that $W(r) = V(r) = 1/r$, and familiar rule follows immediately.

$$V(r) = (1 - e^{-rk})(1/r). \tag{9}$$

This particular combination of rules represents the finite level payment stream as the difference between two consols, one with payments beginning today and one with payments deferred for k periods. (The second consol represents the portion of the first consol that extends beyond the life of the annuity).

As a second application, recall that many cyclical patterns can be approximated as weighted sums of leading and lagging sine waves with differing amplitudes and frequencies.

$$C(t) = \textstyle\sum_i a_i \sin(b_i + c_i t). \tag{10}$$

By combining rule 5 from Table I with rules 1 and 5 from Table II, we can represent the present value of the complex cyclical cash flow as:

$$V(r) = \textstyle\sum_i a_i \exp(ra_i/b_i)c_i/(c_i^2 + r_i^2). \tag{11}$$

Note that, in this example, there is no uncertainty about the future cash flow (or about the future rate of discount). Nevertheless, variation in the cash flow, as measured by the amplitude (a_i) and frequency (b_i) of each sine wave, can have a substantial impact on the present value of the cash flow. In particular, present value is an increasing function of frequency at very low values ($c_i < 1$) but decreases with frequency at higher values.

V. Conclusions

A complete set of combinations and permutations of the rules in Table I and the properties in Table II would obviously be quite lengthy. Yet even that list would be short in relation to the extensive tables of Laplace transforms that have been compiled over the years. Many of the individual entries in these lists are of dubious practical value in finance. Nevertheless, the collection of rules could serve as a valuable reference. At the very least, security analysts should be aware that a broader interpretation exists for the present value problem, and that extensive lists of analytic solutions have been found for the generalized problem.